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A COMBINATION OF THE WEIGHT FUNCTION METHOD AND THE LINE SPRING MODEL: A SURFACE-CRACKED CYLINDRICAL SHELL SUBJECTED TO STRESS GRADIENTS

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Abstract— In this paper, the problem of a part-through crack with arbitrary loads on the crack faces is considered. By introducing the solution of the weight function for a two-dimensional edge crack, combined with the line spring model, the extra loads on crack faces can be represented as two equivalent quantities in terms of two-dimensional stress intensity factors. The problem for a cylindrical shell containing an axial inner surface crack subjected to stress gradients can then be reduced to two integral equations. The model is used to compute the stress intensity factors for four stress distributions—uniform, linear, quadratic and cubic—and the results give very good agreement with those obtained by the finite element method. The numerical results of the calculations for axial inner surface crack initiation in the shell subjected to thermal shock are also presented and discussed.

1. INTRODUCTION

The concept of "weight functions" for two-dimensional elastic crack analysis was first introduced by Bueckner (1970), whose contributions led to what is now a vast literature on two-dimensional elastic crack analysis. Rice (1972) showed that weight functions could be determined by differentiating known elastic displacement field solutions with respect to crack length. Knowledge of the two-dimensional elastic crack solution, as a function of crack length, for any one loading enables one to determine directly the effect of the crack on the elastic solution for the same body under any loading system. In recent years, there has been a surge of interest in the three-dimensional theory, which showed new types of investigations, including crack tip interactions with dislocation and other defects, stress analysis for the perturbed crack shapes, and the configrational stability of the crack shape during growth (Rice, 1989). However, the analytical solutions for weight functions for three-dimensional crack problems are quite limited except for half-plane and circular cracks.

The line spring model (LSM), developed by Rice and Levy (1972), which reduces the three-dimensional part-through crack problems to a problem in thin shell theory, has been proved to be simple and accurate. Delale and Erdogan (1982a,b) modified this model by using Reissner's plate bending theory. Yang (1988) used the method of weight functions to apply the line spring model with classical plate theory to arbitrary loadings. Work similar to but independent of that of Yang (1988), based on the approximate analytical solutions of the stress intensity factors for plane-strain edge-cracked problems given by Wu (1984), was done by one of the authors [see Fan *et al.* (1990, 1992)], and the problem for the thermal shock fracture in a surface-cracked plate was analyzed. The solutions were found to be sensitive to the accuracy of the two-dimensional weight functions.

In this paper, a new, wide range of closed-form weight functions for a single crack in a finite strip, recently derived by Wu (1990, 1991), is introduced as the basis of the analysis. Since the exact solutions of the stress intensity factors and crack displacements at two extremes (relative crack length equal to zero and unity) were used, very good accuracy for the weight function over the entire crack length range was ensured. Combined with the line spring model, the arbitrary loads applied on crack faces for a part-through crack can be

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represented by two equivalent quantities, the equivalent membrane resultant and moment resultant, in terms of two-dimensional stress intensity factors. Then, by using a Reissner type of shell theory, the problem for a cylindrical shell containing a part-through surface crack, subjected to arbitrary loadings, can be reduced to two coupled integral equations in terms of the derivatives of the crack surface displacement and rotation. A cylindrical shell with an axial internal surface crack is solved as an example. The numerical results for four stress distributions—uniform, linear, quadratic and cubic—are compared with those obtained by the finite element method (Raju and Newman, 1979). Finally, the model is used to compute the stress intensity factors in thermal shock loadings, and several results of the calculations are presented and discussed.

2. THE EDGE CRACK WEIGHT FUNCTION

The basic relationship for computing the stress intensity factors due to an arbitrary crack face stress distribution $\sigma_n(X)$, which is defined as the stress at the prospective crack location in the crack-free body, is:

$$K_I = \int_0^A \sigma_n(X) P(A, X)\,\mathrm{d}X, \tag{1}$$

where A and X are the crack length and co-ordinate along the crack, respectively. For edge cracks, we set the origin of the co-ordinates at the crack mouth, and thus $X = A$ at the crack tip. For convenience, we introduce dimensionless quantities as $\sigma_n(x)/\sigma_n$, $\xi = A/h$ and $x = X/h$, where σ_n is the scaling factor with the dimension of stress and h is a characteristic length, here referred to as the shell thickness. Equation (1) can now be rewritten as

$$K_{I0} = \sigma_n h^{1/2} \int_0^{\xi} \frac{\sigma_n(x)}{\sigma_n} p(\xi, x)\,\mathrm{d}x \tag{2}$$

or

$$\left.\begin{aligned} K_{I0} &= \sigma_n h^{1/2} g(\xi) \\ g(\xi) &= \int_0^{\xi} \frac{\sigma_n(x)}{\sigma_n} p(\xi, x)\,\mathrm{d}x, \end{aligned}\right\} \tag{3}$$

where $p(\xi, x)$ is the weight function, which is closely related to the crack face displacement for the reference load case. In deriving $p(\xi, x)$, Wu (1984) first obtained a closed-form weight function that yields good results up to $\xi = 0.6$. Recently, by introducing the exact solutions of the stress intensity factors and crack displacements at two extremes (relative crack length equal to zero and unity), Wu (1990, 1991) presented a wide-range weight function for an edge crack as follows:

$$p(\xi, x) = \frac{1}{\sqrt{2\pi\xi}} \sum_{i=1}^{5} \beta_i(\xi) \left(1 - \frac{x}{\xi}\right)^{i-3/2}, \tag{4}$$

where

$$\left.\begin{aligned}\beta_1(\xi) &= 2\\ \beta_2(\xi) &= [4\xi f_r'(\xi)+2f_r(\xi)+\tfrac{3}{2}\omega_2(\xi)]/f_r(\xi)\\ \beta_3(\xi) &= [\xi\omega_2'(\xi)+\tfrac{5}{2}\omega_3(\xi)-\tfrac{1}{2}\omega_2(\xi)]/f_r(\xi)\\ \beta_4(\xi) &= [\xi\omega_3'(\xi)+\tfrac{7}{2}\omega_4(\xi)-\tfrac{3}{2}\omega_3(\xi)]/f_r(\xi)\\ \beta_4(\xi) &= [\xi\omega_4'(\xi)-\tfrac{5}{2}\omega_4(\xi)]/f_r(\xi)\end{aligned}\right\} \tag{5}$$

and

$$\left.\begin{aligned}\omega_1(\xi) &= 4f_r(\xi)\\ \omega_2(\xi) &= \frac{1}{12\sqrt{2}}[315\pi\chi(\xi)-1057V(\xi)-208\sqrt{2}f_r(\xi)]\\ \omega_3(\xi) &= \frac{1}{30\sqrt{2}}[-1260\pi\chi(\xi)+525V(\xi)+612\sqrt{2}f_r(\xi)]\\ \omega_4(\xi) &= \sqrt{2}V(\xi)-[\omega_1(\xi)+\omega_2(\xi)+\omega_3(\xi)]\\ \chi(\xi) &= \frac{1}{\xi^2}\int_0^{\xi} s^2 f_r^2(s)\,\mathrm{d}s,\end{aligned}\right\} \tag{6}$$

where $f_r(\xi)$ is the stress intensity factor and $V(\xi)$ the crack mouth opening of the uniform load case. The following expressions for f_r and V are used:

$$\left.\begin{aligned}f_r(\xi) &= \Sigma_0^7\, c_n\xi^n/(1-\xi)^{2/3}\\ V(\xi) &= \Sigma_0^7\, v_n\xi^n(1-\xi)^2\end{aligned}\right\} \tag{7}$$

with

c_n: 1.1214, −1.6349, 7.3168, −18.7746, 31.8028, −33.2295, 19.1286, −4.6091;

v_n: 2.9086, −5.5749, 19.5120, −39.0199, 58.2697, −54.7124, 29.4039, −6.8949.

It has been shown by Wu that the accuracy of eqn (7) is within 0.02% of the values obtained by Joseph and Erdogan (1989) and Gregory (1977). Thus, the function $g(\xi)$ in eqn (3) can be obtained by

$$g(\xi) = \frac{1}{\sqrt{2\pi\xi}}\int_0^{\xi}\frac{\sigma_n(x)}{\sigma_n}\sum_{i=1}^{5}\beta_i(\xi)\left(1-\frac{x}{\xi}\right)^{i-3/2}\mathrm{d}x. \tag{8}$$

In the following analysis, the notations $g_t(\xi)$ and $g_b(\xi)$ will be used to denote the function $g(\xi)$ for the uniform stress distribution and pure bending stress cases, respectively.

3. LINE SPRING MODEL WITH ARBITRARY LOADS ON CRACK FACES

The part-through crack geometry for the shell under consideration is shown in Fig. 1. According to the concept of the line spring model, the constitutive relations for the LSM with arbitrary loads on the crack faces can be expressed as [see Fan *et al.* (1990, 1992) and Yang (1988)]

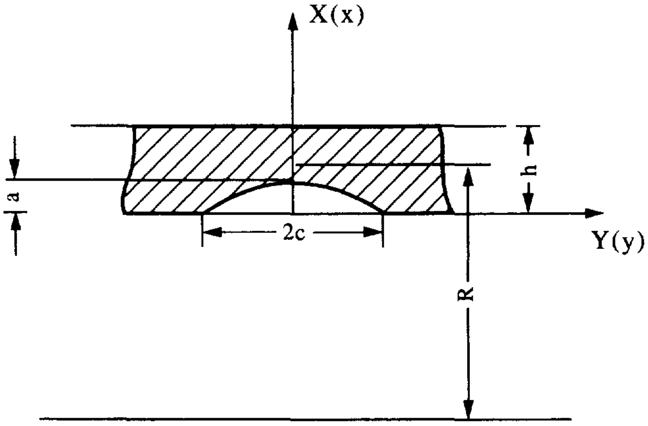

Fig. 1. The geometry of an axial part-through surface crack in a cylindrical shell.

$$\left.\begin{aligned}\delta &= \frac{2(1-\nu^2)h}{E}(\alpha_{tt}\sigma+\alpha_{tb}m+\delta_0)\\ \theta &= \frac{12(1-\nu^2)h}{E}(\alpha_{tb}\sigma+\alpha_{bb}m+\theta_0),\end{aligned}\right\} \tag{9}$$

where σ and m denote, respectively, the net ligament stress and moment per unit length along the cracked section, and δ and θ the relative displacement and rotation of the line springs. The coefficients α_{ij} $(i,j = t, b)$ and (δ_0, θ_0) can be determined from the related plane-strain problem, and are

$$\alpha_{ij} = \int_0^{\xi} g_i(\xi)g_j(\xi)\,\mathrm{d}\xi \quad (i,j = t,b) \tag{10}$$

$$\left.\begin{aligned}\delta_0 &= h^{-1/2}\int_0^{\xi} K_{I0}\, g_t\,\mathrm{d}\xi\\ \theta_0 &= h^{-1/2}\int_0^{\xi} K_{I0}\, g_b\,\mathrm{d}\xi\\ \xi(x) &= l(x)/h.\end{aligned}\right\} \tag{11}$$

For the inner surface crack as shown in Fig. 1, let

$$\delta_0 = 2au(y), \quad \theta_0 = -2\beta_x(y), \tag{12}$$

where $u(y)$ and $\beta_x(y)$ are the dimensionless crack surface opening displacement and rotation, respectively, under the co-ordinate system in Fig. 1. The inverse form of eqn (9) is

$$\left.\begin{aligned}\sigma &= E(\gamma_{tt}u-\gamma_{tb}\beta)-\sigma_0\\ m &= 6E(\gamma_{bt}u-\gamma_{bb}\beta)-m_0\end{aligned}\right\} \tag{13}$$

and

$$
\left.\begin{aligned}
\sigma_0 &= \frac{\alpha_{bb}\delta_0 - \alpha_{bt}\theta_0}{\Delta} \\
m_0 &= \frac{\alpha_{tt}\theta_0 - \alpha_{bt}\delta_0}{\Delta},
\end{aligned}\right\} \tag{14}
$$

where

$$
\left.\begin{aligned}
\gamma_{tt} &= \frac{a}{h(1-v^2)}\frac{\alpha_{bb}}{\Delta}, \quad \gamma_{bb} = \frac{1}{36(1-v^2)}\frac{\alpha_{tt}}{\Delta} \\
\gamma_{tb} &= -\frac{1}{6(1-v^2)}\frac{\alpha_{bt}}{\Delta}, \quad \gamma_{bt} = -\frac{a}{6h(1-v^2)}\frac{\alpha_{tb}}{\Delta}
\end{aligned}\right\} \tag{15}
$$

$$
\Delta = \alpha_{tt}\alpha_{bb} - \alpha_{tb}^2. \tag{16}
$$

From eqn (14), we find that σ_0 and m_0, which are equivalent to the extra loads on the crack faces, can be expressed simply in terms of two-dimensional stress intensity factors. Because σ_0 and m_0 are related to the ratio of crack depth to thickness, ξ, along the crack front, they can be regarded as functions of y, i.e. $\sigma_0 = \sigma_0(y)$, $m_0 = m_0(y)$. Once σ and m are determined, the stress intensity factors along the crack tip front for the part-through crack can be computed from

$$
K_I = K_{I0} + h^{1/2}[\sigma g_t(\xi) + m g_b(\xi)]. \tag{17}
$$

4. INTEGRAL EQUATIONS

In the formulation of the crack problem for the shell, the derivatives of the crack surface displacement and rotation are used as the unknown functions, which are defined by

$$
\frac{\partial}{\partial y}\beta_x(y) = G_1(y), \quad \frac{\partial}{\partial y}u(y) = G_2(y). \tag{18}
$$

It has been shown (Delale and Erdogan, 1979) that the general problem for a symmetrically loaded shell containing an axial through crack may be reduced to the following system of integral equations:

$$
\left.\begin{aligned}
&\frac{1}{2\pi}\int_{-1}^{1}\frac{G_2(\eta)}{\eta-y}\,d\eta - \frac{1}{2\pi}\int_{-1}^{1}[k_{11}(y,\eta)G_1(\eta)+k_{12}(y,\eta)G_2(\eta)]\,d\eta = \sigma/E \\
&\frac{h}{24c\pi}\int_{-1}^{1}\frac{G_2(\eta)}{\eta-y}\,d\eta + \frac{c}{2\pi h}\int_{-1}^{1}[k_{21}(y,\eta)G_1(\eta)+k_{22}(y,\eta)G_2(\eta)]\,d\eta = m/6E
\end{aligned}\right\} \tag{19}
$$

subject to

$$
\int_{-1}^{1} G_1(\eta)\,d\eta = 0, \quad \int_{-1}^{1} G_2(\eta)\,d\eta = 0, \tag{20}
$$

where the kernels $k_{ij}(y,\eta)$ $(i,j = 1,2)$ are known functions [see Delale and Erdogan (1979)], and σ and m denote, respectively, the net ligament stress and moment per unit length along the cracked section. Now, substituting eqn (13) into eqn (19), we obtain

$$\left.\begin{aligned}
&-\gamma_{tt}(y)\int_{-1}^{y} G_1(\eta)\,\mathrm{d}\eta+\gamma_{tb}(y)\int_{-1}^{y} G_2(\eta)\,\mathrm{d}\eta+\frac{1}{2\pi}\int_{-1}^{1}\frac{G_2(\eta)}{\eta-y}\mathrm{d}\eta \\
&\qquad -\frac{1}{2\pi}\int_{-1}^{1}[k_{11}(y,\eta)G_1(\eta)+k_{12}(y,\eta)G_2(\eta)]\,\mathrm{d}\eta = -\frac{\sigma_0}{E} \\
&\gamma_{bt}(y)\int_{-1}^{y} G_1(\eta)\,\mathrm{d}\eta-\gamma_{bb}(y)\int_{-1}^{y} G_2(\eta)\,\mathrm{d}\eta-\frac{c}{24h\pi}\int_{-1}^{1}\frac{G_2(\eta)}{\eta-y}\mathrm{d}\eta \\
&\qquad +\frac{c}{2\pi h}\int_{-1}^{1}[k_{21}(y,\eta)G_1(\eta)+k_{22}(y,\eta)G_2(\eta)]\,\mathrm{d}\eta = \frac{m_0}{6E}.
\end{aligned}\right\} \tag{21}$$

5. SOLUTION FOR A CYLINDRICAL SHELL CONTAINING AN INNER SURFACE CRACK

For an axial semi-elliptical inner surface crack, the crack depth along the Y-axis is given (see Fig. 1) by

$$\xi(y) = a\sqrt{1-y^2}, \quad y = Y/c. \tag{22}$$

The solution of the problem is obtained for four stress distributions on crack faces: uniform, linear, quadratic and cubic. These stresses are symmetrical about the crack face and can be written as

$$(\sigma_n)_j = \sigma_0(X/a)^j \quad (j = 0, 1, 2, 3). \tag{23}$$

For comparison, the Poisson's ratio ν is taken as 0.3. The solution of the integral equations (21) is of the form

$$G_i(\eta) = \frac{\phi_i(\eta)}{(1-\eta^2)^{1/2}} \quad (i = 1, 2) \quad (-1 < \eta < 1), \tag{24}$$

where ϕ_1 and ϕ_2 are bounded functions. The functions ϕ_i may be determined from eqns (21) by using the standard Gauss–Chebyshev integration procedure (Erdogan, 1978). After obtaining ϕ_1 and ϕ_2, the unknowns σ and m representing net ligament stresses may be determined from eqn (13). The stress intensity factor $K_I(y)$ can then be obtained from eqn (17). Let the stress intensity factor be defined as

$$K_I = \sigma_0\sqrt{\frac{\pi a}{Q}}F_j(a/h, a/c, \varphi), \tag{25}$$

where F_j $(j = 1, 2, 3, 4)$ is the coefficient corresponding to $(\sigma_n)_j$ $(j = 0, 1, 2, 3)$, Q is the square of the complete elliptic integral of the second kind and can be approximated by $Q = 1+1.464(a/c)^{1.65}$, and φ is the crack front angle.

The results for F_j at the deepest point of crack penetration, i.e. $\varphi = \pi/2$ $(y = 0)$, are compared with the finite element results of Raju and Newman (1979). Very good agreement

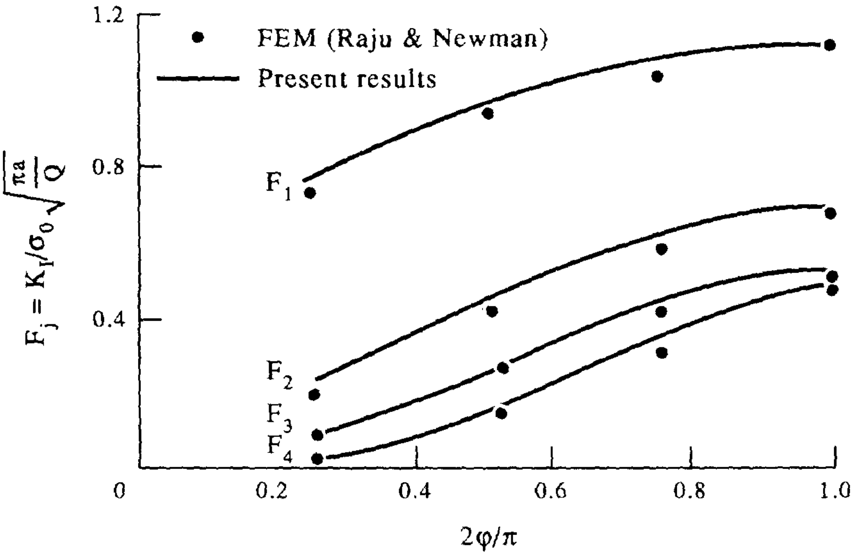

Fig. 2. Comparison of the present results with FEM ($a/c = 0.2$, $a/h = 0.2$).

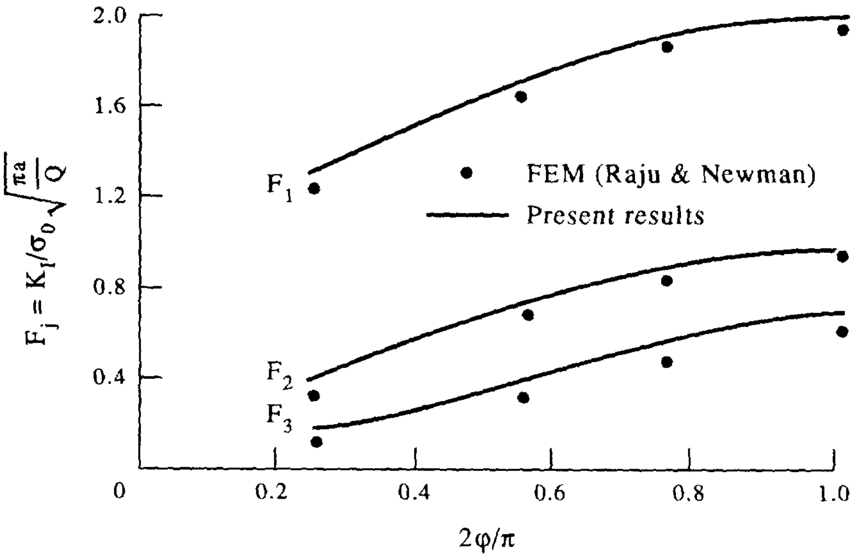

Fig. 3. Comparison of the present results with FEM ($a/c = 0.2$, $a/h = 0.8$).

is found, with a discrepancy of less than 3% up to $a/h = 0.8$. Figures 2–4 give a comparison of the results for the distribution of the stress intensity factors along the crack front. For small values of φ, the solutions given by the line spring model are neither reliable nor meaningful, and these are not presented. It can be seen that, by introducing the wide-range closed-form solutions of the weight functions for an edge crack given by Wu (1990), the accuracy of the surface-cracked solutions is considerably improved, even for large ratios of a/h for the different load cases.

6. STUDIES OF AXIAL INNER SURFACE CRACK INITIATION IN A CYLINDRICAL SHELL SUBJECTED TO THERMAL SHOCK

We consider a cylindrical shell that is initially at a uniform temperature T_0. At time $t > 0$, the temperature at the inner wall is suddenly changed to T_1. In such a thermal shock

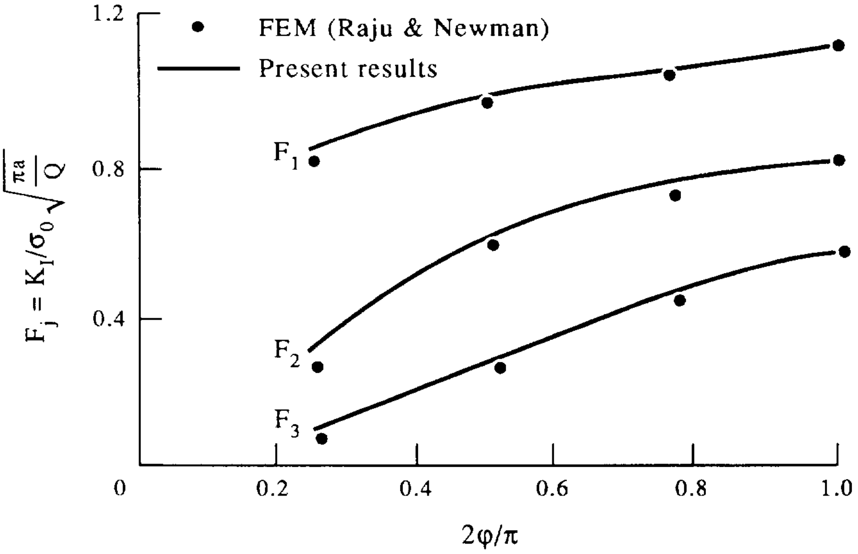

Fig. 4. Comparison of the present results with FEM ($a/c = 0.4$, $a/h = 0.2$).

process, thermal stresses build up in the shell. On introducing a non-dimensional time $\kappa t/R_i^2$, κ being the thermal diffusivity of the shell material and R_i the inner radius of the shell, the transient temperature distribution can be determined by a perturbation method as follows [see Aziz and Na (1984)]:

$$\frac{T-T_0}{T_1-T_0} = \frac{3-r/R_i}{2}\,\mathrm{erfc}\,\frac{r/R_i-1}{2\kappa t/R_i^2}. \tag{26}$$

The hoop thermal stresses in the uncracked shell can be obtained from the quasi-static formulation as

$$\sigma_{\theta\theta} = \frac{\alpha_T E}{1-\nu}\left\{\frac{1+R_i^2/r^2}{R_o^2-R_i^2}\int_{R_i}^{R_o}(T-T_0)r\,\mathrm{d}r + \frac{1}{r^2}\int_{R_i}^{r}(T-T_0)r\,\mathrm{d}r-(T-T_0)\right\}, \tag{27}$$

where R_o is the outer radius of the shell and α_T the coefficient of linear thermal expansion. The inner surface crack problem in the shell can then be formulated by using the stress given by eqn (27). This stress, with the opposite sign, becomes the crack surface tractions in the quasi-static formula. The dimensionless stress intensity factor is introduced as

$$K^* = \frac{1-\nu}{E\alpha_T(T_0-T_1)}K_I\bigg/\sqrt{\frac{\pi a}{Q}}, \tag{28}$$

which, at $\varphi = \pi/2$, is plotted in Fig. 5 for various ratios of crack depth to thickness (with $R_i/R_o = 10/11$). For a statically self-equilibrating stress field, the dimensionless stress intensity factor first increases and then decreases with increasing crack depth. It can be seen that a shallow surface crack is likely to initiate but will not penetrate through the wall under thermal shock conditions only. In addition, the crack initiation cannot take place when $\mathrm{d}K_I/\mathrm{d}t < 0$ during the thermal shock, owing to the inhibiting influence of warm prestressing [see Fan *et al.* (1991)].

Figure 6 shows the dimensionless stress intensity factors as a function of the crack front angle, φ, and the crack depth-to-thickness ratio. For large values of a/h, the maximum

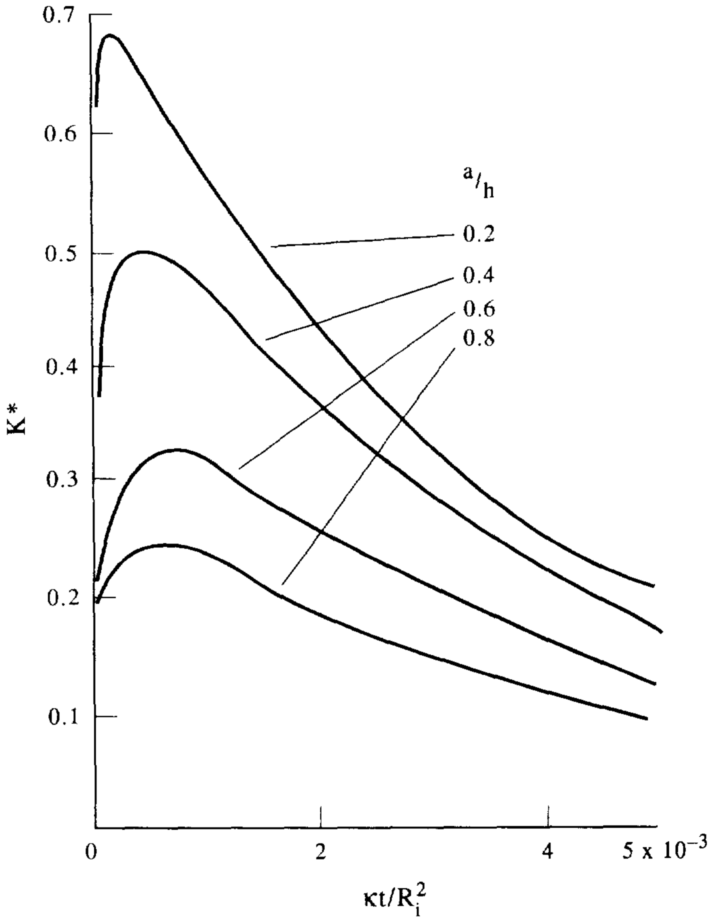

Fig. 5. K^* at $\varphi = \pi/2$ as a function of $\kappa t/R_i^2$ ($R_i/R_o = 10/11$, $a/c = 0.2$).

stress intensity factors may occur near the intersection of the crack with the free surface. In other words, the shallow surface crack may initiate axially prior to the propagation into the wall.

7. CONCLUSIONS

The determination of the stress intensity factor for a surface-cracked body usually requires involved analytical and/or extensive numerical effort, especially when multiple load conditions are to be considered. The method presented here extended the original line spring model to arbitrary loads by introducing accurate solutions of the weight function for two-dimensional analysis. The results obtained show excellent accuracy for a wide range of crack lengths. The model is used to compute the stress intensity factors in a cylindrical shell with an axial inner surface crack subjected to thermal shock. The results show that cracks of small depth are likely to initiate but cannot penetrate through the wall under pure thermal shock conditions. Maximum stress intensity factors may occur at the intersection of the crack with the free surface for a large value of the depth-to-thickness ratio.

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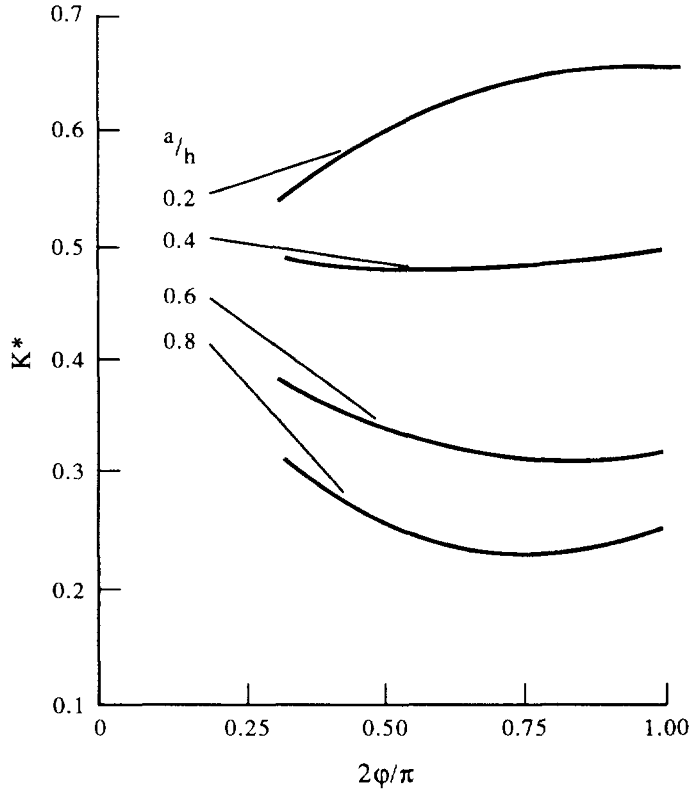

Fig. 6. Distribution of K^* along the crack front ($\kappa t/R_i^2 = 0.0005$, $a/c = 0.2$).

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